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ON THE MERITS OF AN 'APPROXIMATION' TO THE BUSY PERIOD OF THE G--ETC(U)
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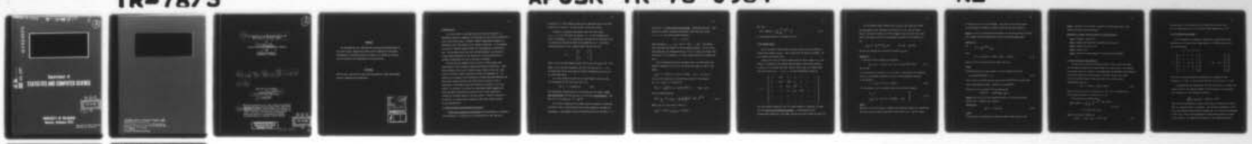

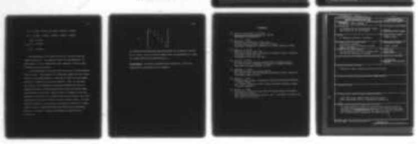

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ON THE MERITS OF AN APPROXIMATION TO THE BUSY PERIOD OF THE GI/G/1 QUEUE .

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ABSTRACT

An approximation for computing the busy period characteristics in the GI/G/1 queue, suggested by D.Gross and C.M.Harris in their book "Fundamentals of Queueing Theory", is shown to be lacking in accuracy, and is therefore quite undesirable for practical use.

KEY WORDS

GI/G/1 queue, busy period, Phase Type distribution, Phase Type Renewal Process, computational probability.

1.Introduction:

In a recent book on queueing theory by Gross and Harris[1], an approximate method is suggested for obtaining the busy period characteristics of the GI/G/1 queue. Through a simple computational example, we demonstrate that this method can be severely inaccurate. It is possible to construct (simple) examples where the approximation is arbitrarily bad. Our discussion will show that the approximation could result in a gross under-estimation of the duration of the busy period and is therefore undesirable for use in practical situations.

We will draw our example from the class of GI/M/1 queues with inter-arrival times following a generalized Erlang distribution. These queues belong to the class of PH/M/1 queues, which in turn are a subclass of the N/G/1 queues, analyzed in detail by Ramaswami [5]. In Section 2, we provide a brief discussion of Phase Type distributions and Phase Type renewal processes and follow this in Section 3 with a discussion of the PH/M/1 queue with particular reference to its busy period. In Section 4 we discuss the approximate method suggested by Gross and Harris obtaining the special form of their approximate formula for the mean length of the busy period. Finally in Section 5, we present our computational example and make some further remarks on the approximation.

2. PH-Distributions and PH-Renewal Processes:

A phase type probability distribution (PH-distribution), introduced by M.F.Neuts[2], is obtained as the distribution of the time till absorption in a finite Markov chain with an absorbing state into which absorption is certain. We shall briefly review such models.

Consider a continuous-time Markov chain with state space $\{1,\dots\dots,n,\ n+1\}$ for which the states $1,\dots\dots,n$ are transient and $(n+1)$ is absorbing. It is assumed that starting in any transient state the chain gets absorbed in $(n+1)$ with probability one. The infinitesimal generator of such a Markov chain then has the form

$$Q = \begin{bmatrix} T & \underline{T}^{o} \\ \underline{0} & 0 \end{bmatrix}$$

where T is an $n\times n$ non-singular matrix with $T_{ii}<0$, and $T_{ij}\geq 0$, $i\neq j$. Also $\underline{T}^{o}\geq \underline{0}$ is an n-vector satisfying $T\underline{e} + \underline{T}^{o} = \underline{0}$, where $\underline{e}'=(1,\dots\dots,1)$. We assume that we are also given an initial probability vector $(\underline{\alpha},\ 0)$ for the Markov chain.

It is now easily seen that for the above Markov chain the time till absorption in the state $(n+1)$ has c.d.f.

$$F(x) = 1 - \underline{\alpha}\exp(Tx)\,\underline{e}, \qquad x \geq 0.$$

Any probability distribution $F(.)$ so obtained will be called a <u>Phase Type Distribution</u>, and the pair $(\underline{\alpha},\ T)$ is called <u>a representation</u>. The mean of $F(.)$ is given by $\lambda^{-1} = -\underline{\alpha}T^{-1}\underline{e}$.

In [3] Neuts considered the renewal process obtained by restarting the Markov chain Q instantaneously after each absorption (renewal) by performing a multinomial trial with probabilities $\underline{\alpha}$ and outcomes $1,\dots,n$

and called it a Phase Type Renewal Process (PH-Renewal Process). This results in another continuous-time Markov chain with state space $\{ 1,\ldots\ldots,n \}$ and infinitesimal generator

$$Q^* = T + T^o A^o,$$

where $A^o = \mathrm{diag}(\alpha_1,\ldots\ldots\ldots,\alpha_n)$ and $T^o = (\underline{T}^o,\ldots\ldots\ldots,\underline{T}^o)$. This Markov chain describes the "phase" of the PH-renewal process. In [2] it has been shown that the representation $(\underline{\alpha}, T)$ of $F(.)$ may be so chosen that the matrix Q^* is irreducible. We shall assume that this is indeed the case.

For the PH-renewal process described above, let $N(t)$ denote the number of renewals in $(0,t]$ and $J(t)$ denote the phase at time $t+$ and define

$$P_{ij}(\nu, t) = P[N(t)=\nu,\ J(t)=j \mid J(0)=i], \quad \nu \geq 0, \quad 1 \leq i,j \leq n.$$

In [3] it is shown that the generating function of the matrices $P(\nu, t) = (P_{ij}(\nu, t))$ is given by

$$\tilde{P}(z, t) = \sum_{\nu=0}^{\infty} z^{\nu} P(\nu, t) = \exp\{(T+zT^oA^o)t\}, \quad |z| \leq 1, \quad t \geq 0.$$

Also by differentiating,

$$M(t) = \frac{\partial}{\partial z} \tilde{P}(z, t)\Big|_{z=1-} = \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{\nu=0}^{k-1} Q^{*\nu} \ T^oA^o \ Q^{*k-1-\nu} \tag{2.1}$$

where the (i, j)-th entry of $M(t)$ is

$$M_{ij}(t) = E[\ N(t)\ I_{[J(t)=j]} \mid J(0)=i].$$

Note that

$$m(t) = \underline{\alpha}M(t)\underline{e} = \underline{\alpha} \sum_{k=1}^{\infty} \frac{t^k}{k!} Q^{*k-1} \underline{T}^{o} \tag{2.2}$$

is the expected number of renewals in (0, t].

3. The PH/M/1 Queue:

We now consider a GI/M/1 queue in which arrivals occur according to a Phase Type renewal process. Such a model will be denoted by PH/M/1. We let μ denote the service rate.

Letting $X(t)$ and $J(t)$ denote respectively the queue length (i.e., the number of customers in the system) and the phase of the arrival process at time t+, it is easily seen that $\{(X(t), J(t)): t \geq 0\}$ is a Markov chain with state space $\{0,1,.....\} \times \{1,....,n\}$ and infinitesimal generator

$$\tilde{Q} = \begin{bmatrix} T & T^oA^o & 0 & 0 & 0 & \cdots & \cdots \\ \mu I & T-\mu I & T^oA^o & 0 & 0 & \cdots & \cdots \\ 0 & \mu I & T-\mu I & T^oA^o & 0 & \cdots & \cdots \\ 0 & 0 & \mu I & T-\mu I & T^oA^o & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

The infinitesimal generator $\tilde{Q}$ is of a form studied by V.Wallace [7] under the name quasi-birth-and-death processes. A queueing model with infinitesimal generator of the same form has also been studied by Neuts [4].

For the Markov chain Q above we let $\tilde{G}_{jj'}(x)$, $x\geq 0$, $1\leq j,j'\leq n$, denote the probability that, starting in the state $(i+1, j)$, $i\geq 0$, the first visit to the set of states $\underline{i} = \{(i,k): 1\leq k\leq n\}$ occurs no later than x and that at the epoch of such a first passage the phase of the system is j'. Let

$$G_{jj'}(s) = \int_0^\infty e^{-sx}\, d\tilde{G}_{jj'}(x)\ , \qquad \text{Re } s\geq 0, \quad 1\leq j,j'\leq n$$

and let $G(s)$ denote the $n\times n$ matrix of entries $G_{jj'}(s)$.

Theorem 3.1

i) The matrix $G(s)$ satisfies the equation

$$G(s) = [sI - (T-\mu I)]^{-1}\,\mu I + [sI - (T-\mu I)]^{-1} T^{o}A^{o}\, G^{2}(s) \tag{3.2}$$

for all $s\geq 0$.

ii) If the queue is stable, i.e., if $\rho=\lambda/\mu<1$, then $G=G(0+)$ is stochastic.

iii) G is the minimal solution in the set of sub-stochastic matrices to the equation

$$G = (\mu I - T)^{-1}\,[\mu I + T^{o}A^{o}G^{2}] \tag{3.3}$$

iv) The matrix G can be computed using the recurrence formulas

$$\left.\begin{aligned} G_0 &= 0 \\ G_{k+1} &= (\mu I - T)^{-1}\,[\mu I + T^{o}A^{o}G_k^{2}]\ , \quad k\geq 0. \end{aligned}\right\} \tag{3.4}$$

Proof:

Equation (3.2) follows from a standard first passage argument by considering the first time the queue length goes either down or up. That the queue is stable iff $\rho=\lambda/\mu<1$ is well-known. The proofs of the other claims in the theorem are similar to those in Neuts[4] and hence omitted.

Remark: In [5] it is shown that the matrix G is strictly positive and thus has a unique invariant probability vector $\underline{g}$ satisfying $\underline{g}G=\underline{g}$, $\underline{g}\underline{e}=1$.

Theorem 3.5:

Let

$$\underline{\mu}^* = -\frac{d}{ds} G(s)\underline{e} \Big|_{s=0+} .$$

Then

$$\underline{\mu}^* = -(I-G+\tilde{G})[(T + T^oA^o) + (T^oA^o - \mu I)\tilde{G}]^{-1}\underline{e}, \tag{3.6}$$

where $\tilde{G}$ is the $n\times n$ matrix each of whose rows is $\underline{g}$.

Proof:

Differentiating (3.2) with respect to s and letting $s\downarrow 0$, we get

$$[(T-\mu I)+T^oA^o+T^oA^oG]\underline{\mu}^* = -\underline{e}. \tag{3.7}$$

Since $I-G+\tilde{G}$ is non-singular and since $\mu I+(T-\mu I)G+T^oA^oG^2=0$, we have that

$$[(T-\mu I)+T^oA^o+T^oA^oG](I-G+\tilde{G}) = (T+T^oA^o) + (T-\mu I+2T^oA^o)\tilde{G}$$

Using this in (3.7) we get (3.6) by noting that

$$(T+T^oA^o)\tilde{G} = (T+T^oA^o)\underline{e}\underline{g} = \underline{0}\underline{g} = 0$$

Remark: Note that μ_i^* is the expected duration of a busy period which starts with 1 customer and in phase i.

Theorem 3.8:

$$\underline{g}\underline{\mu}^* = 1/[\mu(1-\rho)] \tag{3.9}$$

Proof:

This formula is established for the more general N/G/1 queue in [5].

<u>Remark</u>: Equation (3.9) provides a powerful computational check on the accuracy of numerical computations.

<u>Algorithm to compute the mean length of the busy period</u>:

<u>Step 1</u>: Compute G using (3.4)

<u>Step 2</u>: Compute the invariant probability vector $\underline{g}$ of G

<u>Step 3</u>: Compute $\underline{\mu}^*$ using (3.6)

<u>Step 4</u>: Verify (3.9)

<u>Step 5</u>: Compute the mean length of the busy period $E(BP)=\underline{\alpha}\underline{\mu}^*$

4. The Gross-Harris Approximation:

In their book [1], Gross and Harris obtain a "nice, neat" expression for the expected length of the busy period in a GI/G/1 queue. By treating the arrival process as though it were memoryless and mimicking the derivation of the mean length of the busy period in an M/G/1 queue, they "show" that the expected length of the busy period in a GI/G/1 queue is approximately equal to

$$E(BP) \approx E(S)/[1 - \int_0^\infty m(t)\, dB(t)] \tag{4.1}$$

where S denotes the service time and B(.) its c.d.f., and where m(t) is the expected number of arrivals in (0, t].

Note that for the PH/M/1 queue

$$\int_0^\infty m(t)\, dB(t) = \underline{\alpha} \int_0^\infty \sum_{k=1}^{\infty} \frac{t^k}{k!}\, Q^{*k-1}\, \underline{T}^o\, \mu e^{-\mu t} dt \qquad \text{by (2.2)}$$

$$= \mu^{-1}\, \underline{\alpha}(I-\mu^{-1}Q^*)^{-1}\, \underline{T}^o,$$

whence (4.1) may be written as

$$E(BP) \approx 1/[\mu - \underline{\alpha}(I - \mu^{-1}Q^*)^{-1}\, \underline{T}^o] \tag{4.2}$$

The existence of the inverse in (4.2) follows from the fact that Q^* is a stable matrix, i.e., Re $\delta_i<0$ for every eigenvalue δ_i of Q^*.

5. The Computational Example:

For our example we considered a queue with a generalized Erlang distribution for inter-arrival times which, as a PH-distribution, has the representation

$$\underline{\alpha} = (1, 0, 0, 0, 0)$$

$$T = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -10 & 10 & 0 \\ 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix} \quad \text{and} \quad \underline{T}^o = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

The service time distribution was assumed to be exponential with parameter $\mu = 0.483$. It may be verified that for this queue $\rho = 0.900$.

In implementing the algorithm at the end of Section 3 to compute the (exact) value of the expected length of the busy period, we terminated the iterations in (3.4) using the criterion

$$\max_{1 \leq i,j \leq n} |G_{k+1}(i,j) - G_k(i,j)| < 10^{-8}.$$

Then a linear extrapolation, based on the last two iterates, was used to make the computed matrix G stochastic. The invariant probability vector $\underline{g}$ of G was computed using Wachter's method [6]. All these were programmed in APL, and a copy of the program may be obtained by writing to either of the authors. We summarize the results of our computations below:

$$\underline{\mu}^* = (14.55679,\ 20.58772,\ 29.53159,\ 30.85796,\ 32.24840)$$

$$\underline{g} = (\ 0.27996,\ \ 0.54192,\ \ 0.05679,\ \ 0.05938,\ \ 0.06194)$$

$$\underline{g\mu}^* = 20.73940$$

$$\mu^{-1}(1-\rho)^{-1} = 20.73940$$

$$\underline{\alpha}\ \underline{\mu}^* = 14.55679$$

The approximation to the expected length of the busy period was computed using (4.2). The computed value of the "approximation" for E(BP) turned out to be 5.95040 which, when compared to the true value 14.55679, is grossly inaccurate.

A close examination of the model here reveals why the "approximation" fares so poorly. The assumption of a memoryless property for the arrival process in the example amounts to resetting the phase of the PH-renewal process at 1 immediately after every departure. This, for our model, has the effect of "slowing down" the arrivals immediately after each departure and leads to under-estimating the duration of (and the number served during) a busy period. What one needs to recognize is that the resetting of the phase to 1 is repeated several times during a busy period and _not_ just at the end of the first service completion alone. As ρ gets larger and closer to 1, one expects a busy period to consist of a larger number of service completions making the approximation progressively worse as ρ increases. Further by considering a general matrix T of the form

$$T = \begin{bmatrix} -a & a & & & \\ & -a & a & & \\ & & -b & b & \\ & & & -b & b \\ & & & & -b \end{bmatrix}$$

for defining the generalized Erlang distribution and by making 'a' smaller and 'b' larger, one can construct examples where the approximation is worse for queues with even the same value for ρ.

Acknowledgment: We express our gratitude to Professor M. F. Neuts for suggesting this problem and for his guidance.

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